A projection and an effect in a synaptic algebra

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Abstract

We study a pair p, e consisting of a projection p (an idempotent) and an effect e (an element between 0 and 1) in a synaptic algebra (a generalization of the self-adjoint part of a von Neumann algebra). We show that some of Halmos's theory of two projections (or two subspaces), including a version of his CS-decomposition theorem, applies in this setting, and we introduce and study two candidates for a commutator projection for p and e.

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1 Introduction

In [16], P. Halmos studied two projection operators P and Q on a Hilbert space and proved a basic theorem, now called the *CS-decomposition theorem*, that expresses Q in terms of P and positive contraction operators C and S, called the *cosine* and the *sine* operators, respectively, for Q with respect to P. For a lucid and extended exposition of Halmos's theory of two projections,

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see [2]. In [12], we proved a generalization of the CS-decomposition theorem in the setting of a so-called synaptic algebra [12, Theorem 5.6].

In what follows, A is a synaptic algebra with enveloping algebra $R \supseteq A$ [3, 7, 9, 10, 11, 21], P is the orthomodular lattice [1, 18] of projections in A, and E is the convex effect algebra [4, 14] of all effects in A. To help fix ideas, we note that the self-adjoint part of a von Neumann algebra, and more generally of an AW*-algebra, forms a synaptic algebra. Numerous additional examples are given in the literature cited above.

In this article we generalize the CS-decomposition theorem for two projections $p, q \in P \subseteq A$ to the case of a projection $p \in P$ and an effect $e \in E$ (Theorem 3.9 below), and we investigate two candidates for the commutator projection for the pair p and e (Section 5 below).

In our generalization of the CS-decomposition theorem, which we call the *CBS-decomposition theorem*, the cosine and sine effects c and s introduced in [12, Definition 4.2] are generalized (Definition 3.1 below) and supplemented by a third effect b (Definition 3.6 below).

Part of our motivation for the work in this article derives from our interest in *the infimum problem* as applied to the synaptic algebra A, i.e., the problem of determining just when two effects $e, f \in E$ have an infimum $e \wedge f$ in E, and if possible, finding a perspicuous formula for $e \wedge f$ when it does exist. That this problem is non-trivial is indicated by a remark of P. Lahti and M. Mączynski in [19, p. 1674] that the partial order structure of E is "rather wild." The development in [15] and [20] suggests that it might be possible to make progress on the infimum problem for A if the problem can be solved for the pair p, e with $p \in P$ and $e \in E$. We hope that our results in this article will cast some light on the latter problem. In Section 6 below, we illustrate the utility of the CBS-decomposition theorem by applying it to generalize a result of T. Moreland and S. Gudder concerning the infimum problem [20] to the setting of a synaptic algebra.

2 Some basic definitions, notation, and facts

In this section we briefly outline some notions that we shall need below. For the definition of a synaptic algebra and more details, see the literature cited above, especially [3] and [10]. In what follows, the notation := means 'equals by definition,' the ordered field of real numbers and its subfield of rational numbers are denoted by $\mathbb{R}$ and $\mathbb{Q}$, and 'iff' abbreviates 'if and only if.'

The enveloping algebra R of A is a real linear associative algebra and if $a, b \in A$, it is understood that the product ab, which may or may not belong to A, is calculated in R. However, if a commutes with b, in symbols aCb, then $ab = ba \in A$. The *commutant* and *bicommutant* of a are defined and denoted by

$$C(a) := \{b \in A : aCb\} \text{ and } CC(a) := \{c \in A : c \in C(b) \text{ for all } b \in C(a)\},$$

respectively. There is a *unity element* $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

As a subset of R, the synaptic algebra A forms a real linear space which is partially ordered by $\leq$ and for which 1 is a (strong) order unit. If $a, b \in A$ and $a \leq b$, we say that b *dominates* a, or equivalently, that a is a *subelement* of b.

If $a, b, c \in A$, then $ab + ba$, $abc + cba \in A$. Also $aba \in A$ and the *quadratic mapping* $b \mapsto aba$ is linear and order preserving on A.

If $0 \leq a \in A$, there exists a unique *square root*, denoted $a^{1/2} \in A$ such that $0 \leq a^{1/2}$ and $(a^{1/2})^2 = a$; moreover $a^{1/2} \in CC(a)$. Thus, if $0 \leq a$, then $C(a) = C(a^2) = C(a^{1/2})$. If $a \in A$, then $0 \leq a^2$, and the *absolute value* of a is denoted and defined by $|a| := (a^2)^{1/2}$. We note that $|a| \in CC(a)$. The *positive part* of a is denoted and defined by $a^+ := \frac{1}{2}(|a| + a)$. Clearly, $a^+ \in CC(a)$.

Partially ordered by the restriction of $\leq$, the set $P := \{p \in A : p = p^2\}$ of *projections* in A forms an *orthomodular lattice* (OML) [1, 18], [3, §5] with $p \mapsto p^\perp := 1 - p$ as the *orthocomplementation*. The meet (greatest lower bound) and join (least upper bound) of projections $p, q \in P$ are denoted by $p \wedge q$ and $p \vee q$, respectively. The projections $p, q \in P$ are *orthogonal*, in symbols $p \perp q$, iff $p \leq q^\perp$, and it turns out that $p \perp q \Rightarrow p + q = p \vee q$. A minimal nonzero projection in P is called an *atom*. If $p, q \in P$ and p is an atom, then either $p \wedge q = p$ (i.e., $p \leq q$) or else $p \wedge q = 0$.

Calculations in the OML P are facilitated by the following theorem [18, Theorem 5, p. 25].

2.1 Theorem. *For $p, q, r \in P$, if any two of the relations pCq, pCr, or qCr hold, then $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.*

To each element $a \in A$ is associated a unique projection $a^{\text{o}} \in P$ called the *carrier* of a such that, for all $b \in A$, $ab = 0 \Leftrightarrow a^{\text{o}}b = 0 \Leftrightarrow ba^{\text{o}} = 0 \Leftrightarrow ba = 0$. It turns out that $aa^{\text{o}} = a^{\text{o}}a = a$, $a^{\text{o}} \in CC(a)$, $(a^2)^{\text{o}} = a^{\text{o}}$, $|a|^{\text{o}} = a^{\text{o}}$, and if

$p \in P$ and $e \in E$, then $p^{\mathrm{o}} = p$ and $e \leq e^{\mathrm{o}}$. Also, $0 \leq a \leq b \Rightarrow a^{\mathrm{o}} \leq b^{\mathrm{o}}$. Moreover, if $p, q \in P$, then $(pqp)^{\mathrm{o}} = p \wedge (p^{\perp} \vee q)$ [3, Theorem 5.6].

We shall have use for the next two lemmas which follow from [10, Lemma 4.1] and [11, Theorem 5.5].

2.2 Lemma. *If* $0 \leq a_1, a_2, ..., a_n \in A$, *then* $(\sum_{i=1}^{n} a_i)^{\mathrm{o}} = \bigvee_{i=1}^{n} (a_i)^{\mathrm{o}}$.

2.3 Lemma. *If* $a, b, ab \in A$, *then* $(ab)^{\mathrm{o}} = a^{\mathrm{o}} b^{\mathrm{o}} = b^{\mathrm{o}} a^{\mathrm{o}} = a^{\mathrm{o}} \wedge b^{\mathrm{o}}$.

The set $E := \{e \in A : 0 \leq e \leq 1\}$ of *effect elements* (or for short, simply *effects*) in A forms a convex effect algebra [4, 14]. If $e \in E$, then the *orthosupplement of* e is denoted and defined by $e^{\perp} := 1 - e \in E$. Two effects e and f are *disjoint* iff the only effect $g \in E$ with $g \leq e, f$ is $g = 0$. Every projection is an effect, i.e., $P \subseteq E$; in fact, P is the extreme boundary of the convex set E.

2.4 Lemma. *Let* $p, q \in P$. *Then*: (i) *The infimum* $p \wedge q$ *of* p *and* q *in* P *is also the infimum of* p *and* q *in* E. (ii) *The supremum* $p \vee q$ *of* p *and* q *in* P *is also the supremum of* p *and* q *in* E.

Proof. (i) Of course $p \wedge q \leq p, q$, and it remains to prove that if $e \in E$ with $e \leq p, q$, then $e \leq p \wedge q$. But, if $e \leq p, q$, then $e^{\mathrm{o}} \leq p, q$, whence $e \leq e^{\mathrm{o}} \leq p \wedge q$.

(ii) Of course $p, q \leq p \vee q$, and it remains to prove that if $e \in E$ with $p, q \leq e$, then $p \vee q \leq e$. So assume that $p, q \leq e$, and therefore that $e^{\perp} \leq p^{\perp}, q^{\perp}$. It follows that $(e^{\perp})^{\mathrm{o}} \leq p^{\perp}, q^{\perp}$, whence $p, q \leq ((e^{\perp})^{\mathrm{o}})^{\perp} \in P$. Consequently, $p \vee q \leq ((e^{\perp})^{\mathrm{o}})^{\perp}$. But $e^{\perp} \leq (e^{\perp})^{\mathrm{o}}$, so $((e^{\perp})^{\mathrm{o}})^{\perp} \leq e^{\perp\perp} = e$, and we have $p \vee q \leq e$. $\square$

In view of Lemma 2.4, no confusion will result if an existing infimum (respectively, supremum) in E of effects $e, f \in E$ is denoted by $e \wedge f$ (respectively, by $e \vee f$).

By [3, Theorem 2.6 (v)], an effect $e \in E$ is a projection iff e is sharp, i.e., iff e is disjoint from its own orthosupplement $e^{\perp}$ iff $e \wedge e^{\perp} = 0$. Moreover, the carrier e^{o} of an effect $e \in E$ is the smallest projection that dominates e, so E is a *sharply dominating* effect algebra [13].

The next theorem and its corollary provide useful ways to stipulate that a projection p either dominates or is dominated by an effect e.

2.5 Theorem ([3, Theorem 2.4])**.** *Let* $p \in P$ *and* $e \in E$. *Then the following conditions are mutually equivalent*: (i) $e \leq p$. (ii) $e = ep = pe$. (iii) $e = pep$. (iv) $e = ep$. (v) $e = pe$.

2.6 Corollary. *If $p \in P$ and $e \in E$, then the following conditions are mutually equivalent*: (i) $p \leq e$. (ii) $p = ep = pe$. (iii) $p = ep$. (iv) $p = pe$.

Proof. In Theorem 2.5, replace e by $e^{\perp} = 1 - e$ and p by $p^{\perp} = 1 - p$. Then $p \leq e \Leftrightarrow e^{\perp} \leq p^{\perp}$, $1-e = (1-e)(1-p) \Leftrightarrow p = ep$, and $1-e = (1-p)(1-e) \Leftrightarrow p = pe$. ☐

As a consequence of Theorem 2.5 and its corollary, if a projection p and an effect e are comparable (i.e, $e \leq p$ or $p \leq e$), then pCe. One of the reasons that the order structure of E is so "wild" is that the same does not hold for two effects.

2.7 Lemma. *Suppose that $e, f \in E$ and $p \in P$. Then*: (i) *If eCf, then $ef \in E$ and $ef \leq e, f$.* (ii) *If pCf, then $pf = fp = pfp = p \wedge f$, the infimum of p and f in E.*

Proof. (i) Assume that $e, f \in E$ and $ef = fe$. By [3, Lemma 1.5], $0 \leq ef$. Likewise, $0 \leq e, 1 - f$ and $eC(1 - f)$, so $0 \leq e(1 - f) = e - ef$, whence $ef \leq e \leq 1$, so $ef \in E$. By symmetry, $ef \leq f$.

(ii) Suppose that pCf and let $g \in E$ with $g \leq p, f$. By (i), $pf \leq p, f$. Also, by Theorem 2.5, $g = pgp$, and as $g \leq f$, we have $g = pgp \leq pfp = p^2 f = pf$, whence $pf = p \wedge f$. ☐

In part (i) of Lemma 2.7, we note that although $ef = fe \in E$, it is not necessarily the infimum of e and f in E. In fact, P.J. Lahti and M.J. Mączynski [19, page 1675] give an example of an effect operator e on a two-dimensional Hilbert space such that the infimum of the commuting effects e and $e^{\perp} = 1 - e$ does not exist in E.

2.8 Lemma. *Suppose that $e \in A$ with $0 \leq e$. Then*: (i) $e \in E \Rightarrow 0 \leq e^2 \leq e \leq 1 \Rightarrow e^2 \in E$. (ii) $e^2 \leq 1 \Leftrightarrow e \in E$. (iii) $e \in E \Rightarrow e - e^2 = ee^{\perp} \in E$.

Proof. (i) If $e \in E$, then $e^2 \leq e$ by Lemma 2.7 (i).

(ii) Suppose that $e^2 \leq 1$. Then $0 \leq (1 - e)^2 + (1 - e^2) = 2(1 - e)$, so $e \leq 1$, whence $e \in E$. Conversely, if $e \in E$, then by (i), $e^2 \leq e \leq 1$.

(iii) If $e \in E$, then $0 \leq e - e^2 = e(1 - e) = ee^{\perp}$ by (i) and $e - e^2 \leq e \leq 1$, so $e - e^2 \in E$. ☐

Each element $a \in A$ determines and is determined by a one-parameter family of projections $(p_{a,\lambda})_{\lambda \in \mathbb{R}}$ called its *spectral resolution* and defined by $p_{a,\lambda} := 1 - ((a - \lambda 1)^{+})^{\mathrm{o}}$ for all $\lambda \in \mathbb{R}$ [3, Definition 8.2]. See [3, §8], especially

[3, Theorem 8.4] for the basic properties of the spectral resolution. We note that by [3, Theorem 8.10], if $a, b \in A$, then bCa iff $bCp_{a,\lambda}$ for all $\lambda \in \mathbb{R}$.

By [3, Theorem 8.4 (vii)], the spectral resolution $(p_{a,\lambda})_{\lambda\in\mathbb{R}}$ is uniquely determined by the corresponding *rational spectral resolution* $(p_{a,\mu})_{\mu\in\mathbb{Q}}$ according to the formula

$$p_{a,\lambda} = \bigwedge\{p_{a,\mu} : \lambda \leq \mu \in \mathbb{Q}\} \text{ for each } \lambda \in \mathbb{R}.$$

2.9 Remark. If $a \in A$ and $q \in P$, then since commutativity of projections is preserved under the formation of arbitrary existing infima, the formula above implies that qCa iff $qCp_{a,\mu}$ for all $\mu \in \mathbb{Q}$.

Let $q \in P$. Then with the partial order and operations inherited from A, the subset

$$qAq := \{qaq : a \in A\} = \{a \in A : a = qaq\} = \{a \in A : a = qa = aq\} \subseteq A$$

is a synaptic algebra in its own right with unity element q and with qRq as its enveloping algebra [3, Theorem 4.10]. The OML of projections in qAq is $P[0, q] := \{v \in P : v \leq q\}$ with the orthocomplementation $v \mapsto v^{\perp_q} := v^\perp \wedge q$. Likewise, the set of all effects in qAq is $E[0, q] := \{f \in E : f \leq q\}$ with the orthosupplementation $f \mapsto f^{\perp_q} := q - f = (1 - f)q = f^\perp q = qf^\perp = f^\perp \wedge q$ (Lemma 2.7 (ii)). Let $a \in qAq$. Then $|a|$, a^+, a^{o}, and if $0 \leq a$, $a^{1/2}$, belong to qAq and coincide with the absolute value, the positive part, the carrier, and the square root of a, respectively, as calculated in qAq.

2.10 Lemma. *Let $a \in A$, $f \in E$, and $q \in P$. Then*: (i) *If qCa, then the spectral resolution of $qa = aq \in qAq$ as calculated in qAq is given by $(qp_{a,\lambda})_{\lambda\in\mathbb{R}} = (p_{a,\lambda} \wedge q)_{\lambda\in\mathbb{R}}$.* (ii) *If qCf, then the spectral resolution of $qf = fq = f \wedge q \in qAq$, as calculated in qAq, is given by $(qp_{f,\lambda})_{\lambda\in\mathbb{R}} = (p_{f,\lambda} \wedge q)_{\lambda\in\mathbb{R}}$.*

Proof. Part (i) is proved by a direct calculation using [3, Definition 8.2 and Theorem 4.10] and the fact that qCa implies $qCp_{a,\lambda}$, whence $p_{a,\lambda} \wedge q = qp_{a,\lambda}$ for all $\lambda \in \mathbb{R}$. Part (ii) follows from (i) and Lemma 2.7 (ii). □

2.11 Lemma. *Suppose that p is an atom in P. Then*: (i) *$pAp = \{\lambda p : \lambda \in \mathbb{R}\}$.* (ii) *If $a \in A$, there exists a unique $\lambda \in \mathbb{R}$ such that $pap = \lambda p$.* (iii) *If $f \in E$ and $pfp = \lambda p$, then $0 \leq \lambda \leq 1$.*

Proof. (i) Since p is an atom, it follows that 0 and $p \neq 0$ are the only projections in the synaptic algebra pAp, from which, using spectral theory in pAp, (i) follows. Part (ii) follows from the fact that $p \neq 0$, and (iii) is a consequence of $0 \leq f \leq 1 \Rightarrow 0 \leq pfp \leq p1p = p^2 = p \leq 1$. □

An element $u \in A$ is said to be a *symmetry* [10] iff $u^2 = 1$, and a *partial symmetry* is an element $t \in A$ such that $t^2 \in P$. As a consequence of the uniqueness theorem for square roots, a projection is the same thing as a partial symmetry p such that $0 \leq p$. If $t \in A$ is a partial symmetry, then $u := t + (t^2)^\perp$ is a symmetry called the *canonical extension* of t.

If $a \in A$ there is a uniquely determined partial symmetry $t \in A$, called the *signum* of a, such that $t^2 = a^{\text{o}}$ and $a = |a|t$. Moreover, $t \in CC(a)$, $t^{\text{o}} = a^{\text{o}}$, and if $u = t + (t^2)^\perp$ is the canonical extension of t to a symmetry, then $u \in CC(a)$ and $a = |a|u = u|a|$. The latter formula is called the *polar decomposition* of a. It turns out that the symmetry u in the polar decomposition of a is uniquely determined.

If $a, b \in A$ and $u \in A$ is a symmetry, it is not difficult to verify that $a \leq b \Leftrightarrow uau \leq ubu$ and that $ua^{\text{o}}u = (uau)^{\text{o}}$.

Two projections $p, q \in P$ are *exchanged by a symmetry* $u \in A$ iff $upu = q$ (whence, automatically, $uqu = p$) and they are *exchanged by a partial symmetry* $t \in A$ iff $tpt = q$ and $tqt = p$. If p and q are exchanged by a partial symmetry t, then they are exchanged by the canonical extension $u := t + (t^2)^\perp$ of t to a symmetry.

If $p \in P$ and $a \in A$, then by direct calculation using the fact that $p^\perp = 1 - p$, one obtains the well-known *Peirce decomposition* of a with respect to p, namely

$$a = pap + pap^\perp + p^\perp ap + p^\perp ap^\perp.$$

We refer to $pap + p^\perp ap^\perp$ as the *diagonal part* of a with respect to p and to $pap^\perp + p^\perp ap$ as the *off-diagonal part* of a with respect to p. We note that pap, $p^\perp ap^\perp$, and the diagonal part $pap + p^\perp ap^\perp$ of a belong to A. Also, although $pap^\perp$ and $p^\perp ap$ belong to the enveloping algebra R, but not necessarily to A, the off-diagonal part $pap^\perp + p^\perp ap$ belongs to A.

2.12 Lemma ([12, Theorem 2.12])**.** *If $0 \leq a \in A$ and $p \in P$, then $a = 0$ iff the diagonal part of a with respect to p is zero.*

2.13 Lemma. *Let $a \in A$ and $p \in P$. Then the following conditions are mutually equivalent*: (i) pCa. (ii) *The off-diagonal part of a with respect to p is zero.* (iii) $pa \in A$. (iv) $ap \in A$. (v) $pap^\perp = 0$. (vi) $p^\perp ap = 0$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from [12, Theorem 2.12]. If $pa \in A$, then since $pa + ap \in A$, we have $ap = (pa + ap) - pa \in A$; similarly, $ap \in A \Rightarrow pa \in A$, and we have (iii) $\Leftrightarrow$ (iv). To prove that (i) $\Leftrightarrow$ (iii), note

that $pCa \Rightarrow pa = ap \in A$. Conversely, suppose that $pa \in A$. Then, since (iii) $\Leftrightarrow$ (iv), $ap \in A$. Also, $(1-p)pa = 0$, so $pa(1-p) = 0$, and we have $pa = pap$. Similarly, $ap(1-p) = 0$, so $(1-p)ap = 0$, i.e., $ap = pap$, whence $pa = pap = ap$. This proves that (i) $\Leftrightarrow$ (iii), and it follows that conditions (i)–(iv) are mutually equivalent.

If (i) holds, then $pap^\perp = app^\perp = 0$, so (i) $\Rightarrow$ (v). Conversely, if (v) holds, then $0 = pap^\perp = pa(1-p) = pa - pap$, so $pa = pap \in A$, and we have (v) $\Rightarrow$ (iii). Similarly, (i) $\Rightarrow$ (vi) $\Rightarrow$ (iv). $\square$

3 A projection and an effect

3.1 Standing Assumption. *For the remainder of this article we assume that $p \in P$, and $e \in E$.*

In this section we associate with the pair p, e four special effects, c, s, j, and b (Definitions 3.2, 3.4, and 3.6) and a symmetry k (Definition 3.8). Using c, s, j, b, and k, we rewrite the Peirce decomposition of e with respect to p, thus obtaining the *CBS-decomposition of e with respect to p* (Theorem 3.9).

In the next definition we generalize to the present case the definitions of the cosine and sine effects for a projection q with respect to the projection p [12, Definition 4.2].

3.2 Definition. Since $0 \le e, e^\perp$, we have $0 \le pep + p^\perp e^\perp p^\perp$ and $0 \le pe^\perp p + p^\perp ep^\perp$. Thus, we define the *cosine* effect c and the *sine* effect s for e with respect to the projection p as follows:

$$(1)\ c := (pep + p^\perp e^\perp p^\perp)^{1/2}. \quad (2)\ s := (pe^\perp p + p^\perp ep^\perp)^{1/2}.$$

3.3 Lemma. (i) *$c^2 = 1 - p + pe + ep - e$.* (ii) *$s^2 = p - pe - ep + e$.* (iii) *$c^2 + s^2 = 1$.* (iv) *$c^2 p = pc^2 = pep$ and $s^2 p^\perp = p^\perp s^2 = p^\perp ep^\perp$.* (v) *$c, s \in C(p)$ and cCs.* (vi) *$c, s, cs, c^2, s^2, c^2s^2 \in E$, $c^2 \le c$, and $s^2 \le s$.*

Proof. Parts (i) and (ii) follow from straightforward calculations using the facts that $p^\perp = 1 - p$ and $e^\perp = 1 - e$. Obviously, (iii) follows from (i) and (ii).

By (i) we have $c^2 p = p - p + pep + ep - ep = pep$ and $pc^2 = p - p + pe + pep - pe = pep$. Using (ii), a similar calculation yields $s^2 p^\perp = p^\perp s^2 = p^\perp ep^\perp$, and we have (iv).

As $0 \leq c, s$, it follows that $C(c) = C(c^2)$ and $C(s) = C(s^2)$. By (iv), $p \in C(c^2)$ and $p \in C(s^2)$, whence pCc and pCs, and (v) is proved.

We have $0 \leq c, s$ and since $c^2, s^2 \leq c^2 + s^2 = 1$, we have $c^2, s^2 \leq 1$, whence by Lemma 2.8, $c^2 \leq c \in E$, and $s^2 \leq s \in E$. Thus, since $c, s \in E$ and cCs, Lemma 2.7 (i) implies that $cs \in E$, and (vi) is proved. $\square$

As $e \in E$, we have $e^2 \in E$ with $e - e^2 = ee^\perp \in E$ (Lemma 2.8 (iii)), whence $p(e-e^2)p + p^\perp(e-e^2)p^\perp \geq 0$.

3.4 Definition. We define $j \in A$ by

$$j := (p(e-e^2)p + p^\perp(e-e^2)p^\perp)^{1/2},$$

i.e., $0 \leq j$ and j^2 is the diagonal part of $e - e^2 = ee^\perp$ with respect to p.

In the next lemma we obtain an important relation between c^2s^2, the diagonal part j^2 of $e-e^2$ with respect to p, and the square of the off-diagonal part $pep^\perp + p^\perp ep$ of e with respect to p.

3.5 Lemma. $c^2s^2 = (cs)^2 = j^2 + (pep^\perp + p^\perp ep)^2$.

Proof. By parts (i) and (ii) of Lemma 3.3,

$$\begin{aligned} c^2s^2 &= (1-p+pe+ep-e)(p-pe-ep+e) = (p-pe-ep+e)-(p-pe-ep+e)^2 \\ &= p - pe - ep + e - p + pe + pep - pe + pep - pepe - pe^2p + pe^2 \\ &\quad + ep - epe - epep + epe - ep + epe + e^2p - e^2 \\ &= e - e^2 + 2p(e-e^2)p - (e-e^2)p - p(e-e^2) + pe^2p + epe - epep - pepe. \end{aligned} \tag{1}$$

Also,

$$(pep^\perp + p^\perp ep)^2 = pep^\perp ep + p^\perp epep^\perp = pe^2p + epe - epep - pepe \tag{2}$$

and

$$\begin{aligned} j^2 &= p(e-e^2)p + (1-p)(e-e^2)(1-p) \\ &= e - e^2 + 2p(e-e^2)p - (e-e^2)p - p(e-e^2). \end{aligned} \tag{3}$$

Combining Equations (1), (2), and (3), we obtain the desired result. $\square$

3.6 Definition. By Lemma 3.5, $0 \le c^2s^2 - j^2$, which enables us to define

$$b := (c^2s^2 - j^2)^{1/2}.$$

We refer to b as the *commutator effect* for the pair p, e (see Lemma 3.11 below).

3.7 Theorem. (i) *pCj and pCb.* (ii) *$b \in E$.* (iii) *$b = |pep^\perp + p^\perp ep|$.*

Proof. (i) Since

$$p(p(e-e^2)p + p^\perp(e-e^2)p^\perp) = p(e-e^2)p = (p(e-e^2)p + p^\perp(e-e^2)p^\perp)p,$$

we have $pC(p(e-e^2)p + p^\perp(e-e^2)p^\perp)$, and since

$$j = (p(e-e^2)p + p^\perp(e-e^2)p^\perp)^{1/2}$$

it follows that pCj. Also, by Lemma 3.3 (v), $pC(c^2s^2)$, and therefore $pC(c^2s^2 - j^2)$. As $b = (c^2s^2 - j^2)^{1/2}$, it follows that pCb.

(ii) Evidently, $0 \le b$. Also by Lemma 3.3 (vi), $b^2 \le c^2s^2 \le 1$, and it follows from Lemma 2.8 (ii) that $b \in E$.

Part (iii) follows immediately from Lemma 3.5 and Definition 3.6. $\square$

3.8 Definition. As per Theorem 3.7 (iii), we define the symmetry k by polar decomposition of $pep^\perp + p^\perp ep$, so that

$$pep^\perp + p^\perp ep = |pep^\perp + p^\perp ep|k = bk = kb$$

where $k \in CC(pep^\perp + p^\perp ep)$.

3.9 Theorem (CBS-decomposition)**.**

$$e = c^2p + bk + s^2p^\perp, \text{ where}$$

(i) *$pep = c^2p = pc^2$ and $p^\perp ep^\perp = s^2p^\perp = p^\perp s^2$.*

(ii) *$b = |pep^\perp + p^\perp ep| = (c^2s^2 - j^2)^{1/2} \in E$.*

(iii) *k is a symmetry and $pep^\perp + p^\perp ep = bk = kb$.*

(iv) *cCp, sCp, cCs, bCp, and $k \in CC(pep^\perp + p^\perp ep)$.*

(v) *$pbk = bpk = bkp^\perp = pep^\perp$, whence $b(pk - kp^\perp) = b^{\mathrm{o}}(pk - kp^\perp) = 0$.*

(vi) $p^\perp bk = bp^\perp k = bkp = p^\perp ep$.

Proof. Parts (i), (ii), (iii), and the formula $e = c^2p + bk + s^2p^\perp$ follow from Lemma 3.3 (iv), Lemma 3.7 (iii), Definition 3.8, and the Pierce decomposition of e with respect to p. Part (iv) is a consequence of Lemma 3.3 (v), Lemma 3.7 (i), and Definition 3.8.

By (iii) and the fact that bCp, we have $bpk = pbk = p(pep^\perp + p^\perp ep) = pep^\perp = (pep^\perp + p^\perp ep)p^\perp = bkp^\perp$, whence $b(pk - kp^\perp) = pep^\perp - pep^\perp = 0$, proving (v). Part (vi) follows immediately from (v). □

As a consequence of the next lemma, in case e is a projection, then the CBS-decomposition theorem reduces to the generalized CS-decomposition theorem ([12, Theorem 5.6]).

3.10 Lemma. *The following conditions are mutually equivalent*: (i) *e is a projection.* (ii) $j = 0$. (iii) $b = cs$.

Proof. By Lemma 2.12 (i), $e - e^2 = 0$ iff $j = 0$, whence (i) $\Leftrightarrow$ (ii). That (ii) $\Leftrightarrow$ (iii) is an immediate consequence of Definition 3.6. □

3.11 Lemma. *The following conditions are mutually equivalent*: (i) pCe. (ii) $b = 0$. (iii) $b^{\mathrm{o}} = 0$. (iv) $cs = j$. (v) $e = c^2p + s^2p^\perp$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Lemma 2.13 and Theorem 3.7 (iii), and the equivalence (ii) $\Leftrightarrow$ (iii) is obvious. The equivalence (ii) $\Leftrightarrow$ (iv) is a consequence of Definition 3.6, so (i)–(iv) are mutually equivalent. That (ii) $\Rightarrow$ (v) follows from Theorem 3.9, and since p commutes with both c^2 and s^2, it is clear that (v) $\Rightarrow$ (i). □

3.12 Definition. If $a \in A$, $q \in P$, and aCq, then the *component of a in the synaptic algebra qAq* is denoted and defined by $a_q := aq = qa = qaq \in qAq$.

If $a \in A$, $q \in P$, and aCq, it is easy to see that $a = a_q + a_{q^\perp}$ is the unique decomposition of a as a sum of an element in qAq and an element in $q^\perp Aq^\perp$. This decomposition can be useful in deducing properties of a from properties of its components $a_q \in qAq$ and $a_{q^\perp} \in q^\perp Aq^\perp$.

3.13 Lemma. *Let $f \in E$, $q \in P$, and suppose that fCq. Then*: (i) *The component $f_q = fq = qf = f \wedge q$ is an effect in qAq.* (ii) *The orthosupplement of f_q in $E[0, q]$ is the component of $f^\perp$ in qAq, i.e., $f_q^{\perp_q} = qf^\perp = f^\perp q = f^\perp \wedge q = (f^\perp)_q$.*

Proof. By Lemma 2.7 (ii), $qf = fq = f \wedge q \in E[0,q]$, proving (i). Also, $f_q^{\perp_q} = (fq)^{\perp}q = (1-fq)q = q - fq = (1-f)q = f^{\perp}q = f^{\perp} \wedge q = (f^{\perp})_q$, proving (ii). $\square$

3.14 Theorem. *For $p \in P$ and $e \in E$, suppose that $q \in P$ with qCp and qCe. Then*: (i) *q commutes with c, s, b, and k.* (ii) *The cosine, sine, and commutator effects for e_q with respect to $p_q = pq = qp = p \wedge q$ as calculated in qAq are $c_q = cq = qc = c \wedge q$, $s_q = sq = qs = s \wedge q$, and $b_q = bq = qb = b \wedge q$, respectively.* (iii) *The CBS-decomposition of e_q with respect to p_q in qAq is $e_q = c_q^2 p_q + b_q k_q + s_q^2 p_q^{\perp_q} = q(c^2p + bk + s^2p^{\perp}) = (c^2p + bk + s^2p^{\perp})q$.*

Proof. (i) As $c = (pep + p^{\perp}e^{\perp}p^{\perp})^{1/2} \in CC(pep + p^{\perp}e^{\perp}p^{\perp})$, we have qCc and similarly qCs. Likewise, qCb follows from $b = |pep^{\perp} + p^{\perp}ep|$ (Theorem 3.7 (iii)), and qCk follows from $k \in CC(pep^{\perp} + p^{\perp}ep)$.

(ii) Obviously, $p_q e_q p_q = qpep = pepq$. Also, as pCq, we have $p_q^{\perp_q} = p^{\perp} \wedge q = p^{\perp}q = qp^{\perp}$. Moreover, $e_q^{\perp_q} = qe^{\perp} = e^{\perp}q = qe^{\perp}q$. Therefore the cosine effect for e_q with respect to p_q in qAq is

$$(p_q e_q p_q + p_q^{\perp_q} e_q^{\perp_q} p_q^{\perp_q})^{1/2} = (pepq + p^{\perp}e^{\perp}p^{\perp}q)^{1/2} = cq = c_q.$$

Similar computations take care of s_q and b_q. Part (iii) follows from (ii). $\square$

4 Carriers and projection-free effects

The assumptions and notation of Section 3 remain in force. In this section we derive some information about the carriers of the effects e, c, s, j, and b. Also, we introduce two special projections, z and t, associated with the effect e (Definition 4.3 below).

If $f \in E$, $q \in P$, and $q \leq f$, we say that q is a *subprojection* of f; likewise, if $g \in E$ and $g \leq f$, we say that g is a *subeffect* of f.

4.1 Definition. If $f \in E$ and the only subprojection of f is 0, we say that f is *projection free.*

Obviously, every subeffect of a projection-free effect is projection free.

4.2 Lemma. (i) *If $f \in E$, then $((f^{\perp})^{\mathrm{o}})^{\perp}$ is the largest subprojection of f.* (ii) *f is projection free iff $(f^{\perp})^{\mathrm{o}} = 1$.* (iii) *$f^{\perp}$ is projection free iff $f^{\mathrm{o}} = 1$.* (iv) *$f - ((f^{\perp})^{\mathrm{o}})^{\perp}$ and $f^{\perp} - (f^{\mathrm{o}})^{\perp}$ are projection-free effects.*

Proof. Part (i) follows from the fact that $(f^\perp)^{\mathrm{o}}$ is the smallest projection that dominates $f^\perp$ [3, Theorem 2.10 (iv)], and parts (ii) and (iii) are immediate consequences of (i).

(iv) By (i), $((f^\perp)^{\mathrm{o}})^\perp$ is a subprojection of f, so $g := f - ((f^\perp)^{\mathrm{o}})^\perp$ is an effect. We have $g^\perp = 1 - f + ((f^\perp)^{\mathrm{o}})^\perp = f^\perp + ((f^\perp)^{\mathrm{o}})^\perp$, whence by Lemma 2.2, $(g^\perp)^{\mathrm{o}} = (f^\perp)^{\mathrm{o}} \vee ((f^\perp)^{\mathrm{o}})^\perp = 1$, so g is projection free by (ii). Similarly, $f^\perp - (f^{\mathrm{o}})^\perp$ is a projection-free effect. $\square$

4.3 Definition. In what follows, $z := ((e^\perp)^{\mathrm{o}})^\perp$ is the largest subprojection of e and $t := ((e^{\perp\perp})^{\mathrm{o}})^\perp = (e^{\mathrm{o}})^\perp$ is the largest subprojection of $e^\perp$.

We note that $(e^\perp)^{\mathrm{o}} = z^\perp$ and $e^{\mathrm{o}} = t^\perp$. Evidently, $e \in P \Leftrightarrow e = z = t^\perp$.

4.4 Theorem.

(i) *$z, t \in P \cap CC(e)$, $z \leq e \leq e^{\mathrm{o}}$, $t \leq e^\perp \leq (e^\perp)^{\mathrm{o}}$, and $e - z$, $e^\perp - t \in E$.*

(ii) *e is projection free iff $z = 0$ iff $(e^\perp)^{\mathrm{o}} = 1$ and $e^\perp$ is projection free iff $t = 0$ iff $e^{\mathrm{o}} = 1$.*

(iii) *$z \perp t$, i.e., $(e^{\mathrm{o}})^\perp \leq (e^\perp)^{\mathrm{o}}$.*

(iv) *$e - z$ and $e^\perp - t$ are projection-free effects.*

(v) *$(e - z)^{\mathrm{o}} = e^{\mathrm{o}} - z = e^{\mathrm{o}} \wedge z^\perp = t^\perp \wedge z^\perp = (t \vee z)^\perp = (t + z)^\perp$.*

(vi) *$(e^\perp - t)^{\mathrm{o}} = (e - z)^{\mathrm{o}} = (t + z)^\perp$.*

Proof. (i) By [3, Theorem 2.10 (vi)], $z^\perp = (e^\perp)^{\mathrm{o}} \in CC(e^\perp)$, from which $z \in P \cap CC(e)$ follows; similarly, $t \in P \cap CC(e)$.

(ii) Part (ii) follows immediately from Lemma 4.2 (ii).

(iii) Since $e \leq e^{\mathrm{o}}$, it follows that $(e^{\mathrm{o}})^\perp \leq e^\perp$, and therefore $t = (e^{\mathrm{o}})^\perp = ((e^{\mathrm{o}})^\perp)^{\mathrm{o}} \leq (e^\perp)^{\mathrm{o}} = z^\perp$.

(iv) Part (iv) follows immediately from Lemma 4.2 (iv).

(v) We have $e = z + (e - z)$, where $z, e - z \in E$, whence by Lemma 2.2, $e^{\mathrm{o}} = z^{\mathrm{o}} \vee (e-z)^{\mathrm{o}} = z \vee (e-z)^{\mathrm{o}}$. Also, $e - z \leq 1 - z = z^\perp$, whence $(e-z)^{\mathrm{o}} \leq z^\perp$, and it follows that $e^{\mathrm{o}} = z \vee (e - z)^{\mathrm{o}} = z + (e - z)^{\mathrm{o}}$, so $(e - z)^{\mathrm{o}} = e^{\mathrm{o}} - z$. Also, since $z \leq e^{\mathrm{o}}$, we have $e^{\mathrm{o}} - z = e^{\mathrm{o}} \wedge z^\perp = t^\perp \wedge z^\perp$, and the remaining equalities follow from De Morgan and the fact that $z \perp t$.

(vi) Proceeding as in the proof of (v), we have $(e^\perp - t)^{\mathrm{o}} = (e^\perp)^{\mathrm{o}} - t = (e^\perp)^{\mathrm{o}} \wedge t^\perp = z^\perp \wedge t^\perp = t^\perp \wedge z^\perp = (e - z)^{\mathrm{o}} = (t + z)^\perp$. $\square$

4.5 Corollary. (i) $e-e^2=(e-z)-(e-z)^2\leq e-z$. (ii) $e-e^2$ *is projection free.* (iii) $(e-e^2)^{\mathrm{o}}=t^{\perp}\wedge z^{\perp}=(e-z)^{\mathrm{o}}=e^{\mathrm{o}}-z$.

Proof. (i) Since $z\leq e$ and $z\in P$, we have $ze=ez=z$, whence $(e-z)-(e-z)^2=e-z-(e^2-ez-ze+z)=e-e^2$ and $e-e^2\leq e-z$.

(ii) By Theorem 4.4 (iv), $e-z$ is projection free; by part (i), $e-e^2$ is a subeffect of $e-z$; therefore $e-e^2$ is projection free.

(iii) By Lemma 2.3, $(e-e^2)^{\mathrm{o}}=(ee^{\perp})^{\mathrm{o}}=e^{\mathrm{o}}(e^{\perp})^{\mathrm{o}}=t^{\perp}z^{\perp}=t^{\perp}\wedge z^{\perp}$. □

4.6 Theorem.

(i) $c^{\mathrm{o}}=(p\vee z^{\perp})\wedge(p^{\perp}\vee t^{\perp})$ *and* $s^{\mathrm{o}}=(p\vee t^{\perp})\wedge(p^{\perp}\vee z^{\perp})$.

(ii) $(cs)^{\mathrm{o}}=c^{\mathrm{o}}s^{\mathrm{o}}=s^{\mathrm{o}}c^{\mathrm{o}}=c^{\mathrm{o}}\wedge s^{\mathrm{o}}$.

(iii) $(c^2s^2)^{\mathrm{o}}=(cs)^{\mathrm{o}}=(p\vee z^{\perp})\wedge(p\vee t^{\perp})\wedge(p^{\perp}\vee z^{\perp})\wedge(p^{\perp}\vee t^{\perp})$.

(iv) $j^{\mathrm{o}}=(p\vee(t^{\perp}\wedge z^{\perp}))\wedge(p^{\perp}\vee(t^{\perp}\wedge z^{\perp}))$.

(v) $s^{\mathrm{o}\perp}\leq c^2\leq c$ *and* $c^{\mathrm{o}\perp}\leq s^2\leq s$.

(vi) $(s^{\mathrm{o}})^{\perp}e=e(s^{\mathrm{o}})^{\perp}=(s^{\mathrm{o}})^{\perp}\wedge e=(s^{\mathrm{o}})^{\perp}p=p(s^{\mathrm{o}})^{\perp}=(s^{\mathrm{o}})^{\perp}\wedge p$.

(vii) $(c^{\mathrm{o}})^{\perp}e=e(c^{\mathrm{o}})^{\perp}=(c^{\mathrm{o}})^{\perp}\wedge e=(c^{\mathrm{o}})^{\perp}p^{\perp}=p^{\perp}(c^{\mathrm{o}})^{\perp}=(c^{\mathrm{o}})^{\perp}\wedge p^{\perp}$.

Proof. (i) Since $0\leq pqp, p^{\perp}q^{\perp}p^{\perp}$, we infer from [3, Theorem 4.9 (v)] and Lemma 2.2 that

$$\begin{aligned}c^{\mathrm{o}}=[(pep+p^{\perp}e^{\perp}p^{\perp})^{1/2}]^{\mathrm{o}}&=(pep+p^{\perp}e^{\perp}p^{\perp})^{\mathrm{o}}=(pep)^{\mathrm{o}}\vee(p^{\perp}e^{\perp}p^{\perp})^{\mathrm{o}}\\&=(pe^{\mathrm{o}}p)^{\mathrm{o}}\vee(p^{\perp}(e^{\perp})^{\mathrm{o}}p^{\perp})^{\mathrm{o}}=(pt^{\perp}p)^{\mathrm{o}}\vee(p^{\perp}z^{\perp}p^{\perp})^{\mathrm{o}}\\&=[p\wedge(p^{\perp}\vee t^{\perp})]\vee[p^{\perp}\wedge(p\vee z^{\perp})]=[p\wedge(p^{\perp}\vee t^{\perp})]\vee w,\end{aligned}\tag{1}$$

where $w:=p^{\perp}\wedge(p\vee z^{\perp})$. Now $pC(p^{\perp}\vee t^{\perp})$ and pCw, whence

$$[p\wedge(p^{\perp}\vee t^{\perp})]\vee w=(p\vee w)\wedge(p^{\perp}\vee t^{\perp}\vee w).\tag{2}$$

But $pCp^{\perp}$ and $pC(p\vee z^{\perp})$, whence

$$p\vee w=p\vee[p^{\perp}\wedge(p\vee z^{\perp})]=(p\vee p^{\perp})\wedge(p\vee p\vee z^{\perp})=p\vee z^{\perp}.\tag{3}$$

Furthermore, since $w\leq p^{\perp}$,

$$p^{\perp}\vee t^{\perp}\vee w=p^{\perp}\vee t^{\perp}.\tag{4}$$

By Equations (3) and (4),

$$(p \vee w) \wedge (p^{\perp} \vee t^{\perp} \vee w) = (p \vee z^{\perp}) \wedge (p^{\perp} \vee t^{\perp}),$$

whence by Equations (2) and (1), $c^{\mathrm{o}} = (p \vee z^{\perp}) \wedge (p^{\perp} \vee t^{\perp})$. By a similar calculation, $s^{\mathrm{o}} = (p \vee t^{\perp}) \wedge (p^{\perp} \vee z^{\perp})$.

Part (ii) follows from Lemma 2.3, and (iii) follows from (i) and (ii).

To prove (iv), put $q := (e - e^2)^{\mathrm{o}}$, noting that by Corollary 4.5 (iii), $q = t^{\perp} \wedge z^{\perp}$. By Definition 3.4, $j^2 = p(e - e^2)p + p^{\perp}(e - e^2)p^{\perp}$, and again it follows from [3, Theorem 4.9 (v)] and Lemma 2.2 that

$$j^{\,\mathrm{o}} = [p \wedge (p^{\perp} \vee q)] \vee [p^{\perp} \wedge (p \vee q)] = [p \wedge (p^{\perp} \vee q)] \vee v, \tag{5}$$

where $v := p^{\perp} \wedge (p \vee q)$. Now $pC(p^{\perp} \vee q)$ and pCv, whence

$$[p \wedge (p^{\perp} \vee q)] \vee v = (p \vee v) \wedge (p^{\perp} \vee q \vee v). \tag{6}$$

But $pCp^{\perp}$ and $pC(p \vee q)$, so

$$p \vee v = (p \vee p^{\perp}) \wedge (p \vee p \vee q) = p \vee q. \tag{7}$$

Furthermore, since $v \leq p^{\perp}$,

$$p^{\perp} \vee q \vee v = p^{\perp} \vee q. \tag{8}$$

Combining Equations (5)–(8) and the fact that $q = e^{\mathrm{o}} \wedge z^{\perp}$, we obtain (iv).

(v) Since $s^{\mathrm{o}\perp}c^2 = s^{\mathrm{o}\perp}(1 - s^2) = s^{\mathrm{o}\perp} - 0 = s^{\mathrm{o}\perp}$, we have $s^{\mathrm{o}\perp} \leq c^2 \leq c$. Similarly, $c^{\mathrm{o}\perp}s^2 = c^{\mathrm{o}\perp}(1 - c^2) = c^{\mathrm{o}\perp} - 0 = c^{\mathrm{o}\perp}$, whence $c^{\mathrm{o}\perp} \leq s^2 \leq s$.

(vi) Since sCp, we have $(s^{\mathrm{o}})^{\perp}Cp$. Moreover, $(s^{\mathrm{o}})^{\perp}c^2 = (s^{\mathrm{o}})^{\perp}(1 - s^2) = (s^{\mathrm{o}})^{\perp}$; by (v), $b^{\mathrm{o}} \leq s^{\mathrm{o}}$, so $(s^{\mathrm{o}})^{\perp}b = 0$; and $(s^{\mathrm{o}})^{\perp}s^2p^{\perp} = 0$; whence $(s^{\mathrm{o}})^{\perp}e = (s^{\mathrm{o}})^{\perp}(c^2p + bk + s^2p^{\perp}) = (s^{\mathrm{o}})^{\perp}p = (s^{\mathrm{o}})^{\perp} \wedge p$. Similarly, $e(s^{\mathrm{o}})^{\perp} = (pc^2 + kb + p^{\perp}s^2)(s^{\mathrm{o}})^{\perp} = p(s^{\mathrm{o}})^{\perp} = p \wedge (s^{\mathrm{o}})^{\perp}$, so $(s^{\mathrm{o}})^{\perp}Ce$ and $(s^{\mathrm{o}})^{\perp}e = (s^{\mathrm{o}})^{\perp} \wedge e$ by Lemma 2.7(ii). The proof of (vii) is similar. □

4.7 Corollary. *If both e and $e^{\perp}$ are projection free, then $c^{\mathrm{o}} = s^{\mathrm{o}} = d^{\mathrm{o}} = 1$.*

A reasonable formula for b^{o} seems to be elusive; however, we do have partial results as per the following lemma. (Also, see Theorem 5.19 below.)

4.8 Lemma. *Let $v := kpk$. Then*: (i) *v is a projection, the symmetry k exchanges p and v, bCv, and $b^{\mathrm{o}} \leq (p \wedge v^{\perp}) \vee (p^{\perp} \wedge v) = (p \wedge v^{\perp}) + (p^{\perp} \wedge v)$.* (ii) *If p is an atom and $pe \neq ep$, then $p \perp v$ and $b^{\mathrm{o}} = p \vee v = p + v$.* (iii) *If p is an atom and $pe \neq ep$, then there exists $\beta \in \mathbb{R}$ with $b = \beta b^{\mathrm{o}}$, $0 < \beta \leq 1$.*

Proof. (i) Obviously, v is a projection and k exchanges p and v. By parts (iii) and (iv) of Theorem 3.9, bCk and bCp, so bCv. Moreover, by Theorem 3.9 (v), $bpkp = bkp^{\perp}p = 0$, whence, since $v = kpk \in P$,

$$b^{\mathrm{o}} \le ((pkp)^{\mathrm{o}})^{\perp} = (((pkp)^2)^{\mathrm{o}})^{\perp} = ((p(kpk)p)^{\mathrm{o}})^{\perp}$$

$$= ((pvp)^{\mathrm{o}})^{\perp} = (p \wedge (p^{\perp} \vee v))^{\perp} = p^{\perp} \vee (p \wedge v^{\perp}). \tag{1}$$

Starting with the observation that $bp^{\perp}kp^{\perp} = bkpp^{\perp} = 0$, and arguing as above, we deduce that

$$b^{\mathrm{o}} \le p \vee (p^{\perp} \wedge v). \tag{2}$$

By (1) and (2),

$$b^{\mathrm{o}} \le [p^{\perp} \vee (p \wedge v^{\perp})] \wedge [p \vee (p^{\perp} \wedge v)],$$

and using Theorem 2.1 to simplify the right side of the latter inequality, we obtain (i).

(ii) Suppose that p is an atom and $pe \ne ep$. Since k exchanges p and v, it follows that v is also an atom. By Lemma 3.11, $b^{\mathrm{o}} \ne 0$, whence by (i), at least one of the conditions $p \wedge v^{\perp} \ne 0$ or $p^{\perp} \wedge v \ne 0$ must hold. Since p and v are atoms, we have $p \perp v$ in either case, whence $p \wedge v^{\perp} = p$, $p^{\perp} \wedge v = v$, so $p \perp v$ and by (i),

$$0 \ne b^{\mathrm{o}} \le p \vee v = p + v. \tag{3}$$

We claim that $p \le b^{\mathrm{o}}$. Suppose not. Then, since p is an atom, $b^{\mathrm{o}} \wedge p = 0$. Thus, as bCp, we have $b^{\mathrm{o}}Cp$, whence $b^{\mathrm{o}}p = b^{\mathrm{o}} \wedge p = 0$ and it follows that $bp = pb = 0$. Consequently, by Theorem 3.9 (vi), $0 = bpk = pep^{\perp}$, and it follows from Lemma 2.13 that pCe, contradicting $pe \ne ep$. Therefore, $p \le b^{\mathrm{o}}$.

We claim that $v \le b^{\mathrm{o}}$. Suppose not. Then since v is an atom, $b^{\mathrm{o}} \wedge v = 0$. Thus, as bCv, we have $b^{\mathrm{o}}Cv$, whence $b^{\mathrm{o}}v = b^{\mathrm{o}} \wedge v = 0$, and it follows that $bv = vb = 0$. By Theorem 3.9 (vi), $bkp = bp^{\perp}k$, and we have

$$0 = bv = bkpk = bp^{\perp}k^2 = bp^{\perp} = b(1-p) = b - bp, \text{ so } b = bp. \tag{4}$$

By Theorem 3.9 (vi) again, $pep^{\perp} = bpk$ and $p^{\perp}ep = bkp$, whence by (4),

$$pep^{\perp} = bpk = bk \text{ and therefore } p^{\perp}ep = bkp = (pep^{\perp})p = 0,$$

and again it follows from Lemma 2.13 that pCe, contradicting $pe \ne ep$. Therefore, $v \le b^{\mathrm{o}}$.

Now we have $p, v \le b^{\mathrm{o}}$, whereupon $p + v = p \vee v \le b^{\mathrm{o}}$, which together with (3) yields $b^{\mathrm{o}} = p \vee v = p + v$.

(iii) Assume the hypotheses of (iii). By Theorem 3.7 (i), $bp = pb = pbp$ and by Lemma 2.11 (ii), (iii), $pb = bp = pbp = \beta p$ with $0 \leq \beta \leq 1$. Moreover, $bk = kb$ by Theorem 3.9 (iii), and by Theorem 3.9 (v), $pbk = bpk = bkp^{\perp}$. Multiplying both sides of $b = bp + bp^{\perp}$ by k, we obtain $kb = kbp + kbp^{\perp} = kbp + bpk = \beta kp + \beta pk = \beta(kp + pk)$. Multiplying by k again, we get $b = \beta(p + kpk) = \beta(p + v) = \beta b^{\mathrm{o}}$ by (ii). Finally, since $pe \neq ep$, we have $b \neq 0$ by Lemma 3.11, whence $0 < \beta$. $\square$

5 Two commutators

The assumptions and notation set forth above remain in force. In this section we study two candidates for a *commutator projection* for the pair $p \in P$, $e \in E$. Recall that in [12, Definition 2.3] the *Marsden commutator* of two projections $p, q \in P$ is denoted and defined by

$$[p, q] := (p \vee q) \wedge (p \vee q^{\perp}) \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp})$$

and has the property that $pCq \Leftrightarrow [p, q] = 0$. With this in mind, for a projection $w \in P$ to be regarded as a *commutator* for the pair p, e, we shall require—at least—that $pCe \Leftrightarrow w = 0$. (Observe that the commutators defined in [22, §5.1] satisfy the dual condition that commutativity obtains iff the commutator equals 1.)

The simplest candidate for a commutator projection for p and e is the carrier projection b^{o} of the commutator effect b. By Lemma 3.11, b^{o} satisfies our basic condition $pCe \Leftrightarrow b^{\mathrm{o}} = 0$.

5.1 Remark. If it happens that $e \in P$, then $z = e$, $t = e^{\perp}$, and $b = cs$, whence by Theorem 4.4 (iii),

$$b^{\mathrm{o}} = (cs)^{\mathrm{o}} = (p \vee e) \wedge (p \vee e^{\perp}) \wedge (p^{\perp} \vee e) \wedge (p^{\perp} \vee e^{\perp})$$

is the Marsden commutator $[p, e]$ of the pair of projections p and e.

Two projections are in so-called *generic position* [12, Definition 2.1] iff their Marsden commutator is 1; hence, by analogy, we say that the projection p and the effect e are in *generic position* iff $b^{\mathrm{o}} = 1$.

5.2 Theorem. *Suppose that p and e are in generic position. Then*:

(i) $(cs)^{\mathrm{o}} = c^{\mathrm{o}} = s^{\mathrm{o}} = 1$.

(ii) $p \wedge z = p \wedge t = p^\perp \wedge z = p^\perp \wedge t = 0$.

(iii) *The symmetry k in the CBS-decomposition of e with respect to p exchanges the projections p and $p^\perp$.*

Proof. Assume that p and e are in generic position, i.e., $b^{\mathrm{o}} = 1$. Since $b^2 = c^2s^2 - d^2 \leq c^2s^2$, it follows that $1 = b^{\mathrm{o}} = (b^2)^{\mathrm{o}} \leq (c^2s^2)^{\mathrm{o}} = (cs)^{\mathrm{o}} = c^{\mathrm{o}}s^{\mathrm{o}}$, proving (i). Part (ii) follows from (i), Theorem 4.6 (iii), and De Morgan. By Theorem 3.9 (v), $pk = kp^\perp$, whence $kpk = p^\perp$, proving (iii). $\square$

There are two possible shortcomings of b^{o} as a commutator projection for the pair p and e: First, although p commutes with b^{o}, in general, e fails to commute with b^{o} (see Example 5.23 below). Second, as we mentioned earlier, obtaining a perspicuous formula for b^{o} in terms of $e^{\mathrm{o}}, (e^\perp)^{\mathrm{o}}, p, c^{\mathrm{o}}, s^{\mathrm{o}}, z$, t, and k seems to offer a challenge.

In the following definition, we shall extend the Marsden commutator for two projections to a commutator $[F]$ for a finite set $F \subseteq P$ of projections. We note that this definition is dual to [22, Definition 5.1.4], i.e., suprema and infima have been interchanged.

5.3 Definition. Suppose that $F = \{w_1, w_2, \ldots, w_n\} \subseteq P$ is a finite set of projections. For any $w \in P$, let us write $w^1 := w$ and $w^{-1} := w^\perp$. Further, let $D := \{1, -1\}$. Then, as $d = (d_1, d_2, \ldots, d_n)$ runs through D^n, the *commutator* of the set F is denoted and defined by

$$[F] := \bigwedge_{d \in D^n} (w_1^{d_1} \vee w_2^{d_2} \vee \cdots \vee w_n^{d_n}) \in P.$$

Also, we define $[\emptyset] := 0$.

Clearly, $[\{w_1\}] = 0$. Also, if $F = \{w_1, w_2\}$, then

$$[F] = (w_1 \vee w_2) \wedge (w_1^\perp \vee w_2) \wedge (w_1 \vee w_2^\perp) \wedge (w_1^\perp \vee w_2^\perp)$$

is the Marsden commutator of w_1 and w_2. We note that if the special projections 0 or 1 are present in F, then $[F \setminus \{0, 1\}] = [F]$.

5.4 Remark. Suppose that F is a finite subset of P, $q \in P$, and qCw for every $w \in F$. Then since commutativity is preserved under formation of orthocomplements, finite suprema, and finite infima, it follows that $qC[F]$.

5.5 Remark. If F is a finite subset of P, it is obvious that $[F]$ is unchanged if one of the projections in F is replaced by its orthocomplement. As a consequence, if both $w \in F$ and $w^\perp \in F$, then $w^\perp$ can be omitted from F without affecting the value of $[F]$.

By dualizing [22, Theorem 5.1.5 and Prop. 5.1.8], we obtain the following characterization of $[F]$.

5.6 Lemma. *Let $F \subseteq P$ be a finite set of projections and put $r := [F]$. Then*:

(i) $w \in F \Rightarrow rCw$.

(ii) *The projections in the set $\{w \wedge r^\perp : w \in F\}$ commute pairwise.*

(iii) *r is the smallest projection that satisfies* (i) *and* (ii).

(iv) *$r = 0$ iff the projections in the set F commute pairwise.*

Now, by dualizing [22, Def. 5.1.6], we shall extend Definition 5.3 to arbitrary countable subsets W of P. (By *countable*, we mean finite or countably infinite.) However our definition will require that the OML P is *σ-complete*, i.e., that every countable subset of P has a supremum (whence also an infimum) in P. It is known that P is σ-complete iff it is σ-orthocomplete, i.e., iff every countable and pairwise orthogonal subset of P has a supremum in P [17, Corollary 3.4]. According to the discussion in [3, §6], every *generalized Hermitian algebra* [5, 6, 8] is a synaptic algebra with a σ-complete projection lattice. For instance, the self-adjoint part of a von Neumann algebra has a σ-complete (and in fact, a complete) projection lattice. Thus we make the following assumption.

5.7 Standing Assumption. *Henceforth in this section, we assume that the OML P is σ-orthocomplete; hence σ-complete.*

5.8 Remarks. Since there are only countably many finite subsets of a countable set, the supremum in the following definition exists. Also, if $W \subseteq P$ is a finite set, then (as is easily seen) $[W] = \bigvee\{\,[F] : F \subseteq W\}$. Therefore, the following definition provides a true generalization of $[F]$ for a finite set $F \subseteq P$.

5.9 Definition. For an arbitrary countable subset $W \subseteq P$, the *commutator* of W is denoted and defined by

$$[W] = \bigvee\{\,[F] : F \subseteq W \text{ and } F \text{ is finite}\}.$$

5.10 Remark. Suppose that W is a countable subset of P, $q \in P$, and qCw for every $w \in W$. Then since commutativity is preserved under formation of arbitrary existing suprema, it follows from Remark 5.4 that $qC[W]$.

5.11 Remarks. If W is a countable subset of P, then as a consequence of Remark 5.5, $[W]$ is unchanged if one of the projections in W is replaced by its orthocomplement. As a consequence, if both $w \in W$ and $w^\perp \in W$, then $w^\perp$ can be omitted from W without affecting the value of $[W]$.

By dualizing [22, Theorem 5.1.7 and Prop. 5.1.8], we obtain the following characterization of $[W]$.

5.12 Theorem. *If $W \subseteq P$, W is countable, and $r := [W]$, then*:

(i) $w \in W \Rightarrow rCw$.

(ii) *The projections in the set $\{w \wedge r^\perp : w \in W\}$ commute pairwise.*

(iii) *r is the smallest projection with properties* (i) *and* (ii).

(iv) *$r = 0$ iff the projections in the set W commute pairwise.*

Using Assumption 5.7, Definition 5.9, and the notion of a rational spectral resolution, we are now in a position to define an alternative $[p, e]$ to b^{o} as a commutator for the pair p, e.

5.13 Definition. For $p \in P$ and $e \in E$, the *commutator* of the pair p, e is denoted and defined by

$$[p, e] := [\{p\} \cup \{p_{e,\mu} : \mu \in \mathbb{Q}\}]\,.$$

As we shall see in Corollary 5.21 (ii) below, no notational conflict with the Marsden commutator of two projections in [12] will result from the use of the notation $[p, e]$ in Definition 5.13.

We note that, in Definition 5.13, only the *set* of projections in the rational spectral resolution of e is involved—the labeling of these projections by rational numbers plays no role in the computation of $[p, e]$.

In the following theorem, which characterizes $[p, e]$, recall that by Lemma 2.7 (ii), if $q \in P$ and qCe, then $q^\perp e = eq^\perp = e \wedge q^\perp$, the infimum of e and $q^\perp$ in E.

5.14 Theorem. *If $p \in P$ and $e \in E$, then $[p, e]$ is the smallest projection $q \in P$ such that qCp, qCe, and $(p \wedge q^{\perp})C(e \wedge q^{\perp})$.*

Proof. Put $W := \{p\} \cup \{p_{e,\mu} : \mu \in \mathbb{Q}\}$ and $r := [p, e] = [W]$. By Theorem 5.12, we have: (i) $w \in W \Rightarrow rCw$. (ii) The projections in the set $\{w \wedge r^{\perp} : w \in W\}$ commute pairwise. (iii) r is the smallest projection with properties (i) and (ii).

We claim that (iv) rCp, (v) rCe, and (vi) $(p \wedge r^{\perp})C(e \wedge r^{\perp})$. Indeed, since $p \in W$, (i) implies that rCp. Also by (i), for every $\mu \in \mathbb{Q}$, $rCp_{e,\mu}$, whence by Remark 2.9, rCe. Moreover, for every $\mu \in \mathbb{Q}$, we have both $p \in W$ and $p_{e,\mu} \in W$, whence $(p \wedge r^{\perp})C(p_{e,\mu} \wedge r^{\perp})$ by (ii). But by Lemma 2.10, $(p_{e,\lambda} \wedge r^{\perp})_{\lambda \in \mathbb{R}}$ is the spectral resolution of $e \wedge r^{\perp}$ as calculated in $r^{\perp}Ar^{\perp}$; hence by Remark 2.9 again, $p \wedge r^{\perp}$ commutes with $e \wedge r^{\perp}$ in $r^{\perp}Ar^{\perp}$, and therefore also in A. Thus we have (iv), (v), and (vi).

Now assume that $v \in P$, vCp, vCe, and $(p \wedge v^{\perp})C(e \wedge v^{\perp})$. We have to prove that $r \leq v$. By (iii) it will be sufficient to show that (i′) $w \in W \Rightarrow vCw$ and (ii′) the projections in the set $\{w \wedge v^{\perp} : w \in W\}$ commute pairwise. To prove (i′), suppose $w \in W$. If $w = p$, we have vCw, so we can assume that $w = p_{e,\mu}$ for some $\mu \in \mathbb{Q}$. But since vCe, it follows that $vCp_{e,\mu}$, and we have (i′).

To prove (ii′), suppose that $w, q \in W$. First we consider the case $w = p$ and $q = p_{e,\nu}$ with $\nu \in \mathbb{Q}$. Since vCe, we have $eCv^{\perp}$, whence by Lemma 2.10 (ii), $(p_{e,\lambda} \wedge v^{\perp})_{\lambda \in \mathbb{R}}$ is the spectral resolution of $e \wedge v^{\perp}$ as calculated in $v^{\perp}Av^{\perp}$. By hypothesis, $(p \wedge v^{\perp})C(e \wedge v^{\perp})$, and it follows that $(p \wedge v^{\perp})C(p_{e,\nu} \wedge v^{\perp})$. This reduces our argument to the case $w = p_{e,\mu}$ and $q = p_{e,\nu}$ with $\mu, \nu \in \mathbb{Q}$. But, the projections in a spectral resolution commute pairwise, whence $(p_{e,\mu} \wedge v^{\perp})C(p_{e,\nu} \wedge v^{\perp})$, proving (ii′). □

By the following corollary to Theorem 5.14, $[p, e]$ qualifies as a commutator of p and e.

5.15 Corollary. *If $p \in P$ and $e \in E$, then $pCe \Leftrightarrow [p, e] = 0$.*

Proof. If pCe, then $0Cp$, $0Ce$, and $(p \wedge 0^{\perp})C(e \wedge 0^{\perp})$, whence $[p, e] \leq 0$, i.e., $[p, e] = 0$. Conversely, if $[p, e] = 0$, then $(p \wedge 0^{\perp})C(e \wedge 0^{\perp})$, i.e., pCe. □

5.16 Lemma. *Let $r := [p, e]$. Then:* (i) *In the CBS-decomposition of e with respect to p, we have rCp, rCe, rCc, rCs, rCj, rCb and rCk.* (ii) *If $q \in P$, qCp, and qCe, then qCr.*

Proof. (i) By Theorem 5.14, rCp and rCe. Since $c = (pep + p^\perp e^\perp p^\perp)^{1/2} \in CC(pep + p^\perp e^\perp p^\perp)$, it follows that rCc, and similarly, rCs. Also, $rC(e - e^2)$, and because $j = (p(e-e^2)p + p^\perp(e-e^2)p^\perp)^{1/2}$, it follows that rCj. Therefore, as $b = (c^2s^2 - j^2)^{1/2}$, we have rCb. Finally, rCk follows from $k \in CC(pep^\perp + p^\perp ep)$.

(ii) Suppose $q \in P$, qCp, and qCe. Then $qCp_{e,\mu}$ for all $\mu \in \mathbb{Q}$, whence qCr by Definition 5.13 and Remark 5.10. □

5.17 Theorem. *Let $q \in P$, suppose that qCp and qCe, let $r := [p, e]$ and let $v \in P[0, q]$ be the commutator $[p_q, e_q]_{qAq}$ of $p_q = pq$ and $e_q = eq$ as calculated in qAq. Then qCr, pCr, eCr, qCv, pCv, eCv, and $v = r_q = rq = qr = q \wedge r$.*

Proof. Since qCp and qCe, we have qCr by Lemma 5.16 (ii). Also, pCr and eCr by Lemma 5.16 (i). As $v \in P[0, q]$, we have $v = qv = vq$ and $v^{\perp_q} = qv^\perp = v^\perp q$. Thus, by Theorem 5.14 applied to p_q and e_q in the synaptic algebra qAq, we infer that v is the smallest projection in $P[0, q]$ such that

$$\text{(i) } vC(pq), \text{ (ii) } vC(eq), \text{ and (iii) } ((pq) \wedge (v^\perp q))C((eq) \wedge (v^\perp q)).$$

Since $vC(pq)$ and qCp, it follows that $pv = p(qv) = (pq)v = v(pq) = v(qp) = (vq)p = vp$, whence pCv. Likewise, since $vC(eq)$ and qCe, it follows that $ev = e(qv) = (eq)v = v(eq) = v(qe) = (vq)e = ve$, whence eCv. Thus the three elements p, q, and $v^\perp$ commute in pairs, and so do the three elements e, q and $v^\perp$. Consequently, $p \wedge (v \vee q^\perp)^\perp = p \wedge v^\perp \wedge q = pv^\perp q = pqv^\perp q = (pq) \wedge (v^\perp q)$, similarly $e \wedge (v \vee q^\perp) = (eq) \wedge (v^\perp q)$, and we can rewrite (iii) as $(p \wedge (v \vee q^\perp)^\perp)C(e \wedge (v \vee q^\perp)^\perp)$. Furthermore, $(v \vee q^\perp)Cp$ and $(v \vee q^\perp)Ce$, and it follows from Theorem 5.14 that $r = [p, e] \le v \vee q^\perp$. Therefore, $r_q = rq = r \wedge q \le (v \vee q^\perp) \wedge q = v \wedge q = v$.

To complete the proof, we have to show that $v \le r_q$, i.e., that $v \le rq$. Since rCp, rCq, and qCp, we have $(rq)C(pq)$. Likewise, since rCe, rCq, and qCe, we have $(rq)C(eq)$. Thus, with v replaced by rq, conditions (i) and (ii) hold; hence, to prove that $v \le rq$, it will be sufficient to prove that condition (iii) holds with v replaced by rq, i.e., that $((pq) \wedge ((rq)^\perp q))C((eq) \wedge ((rq)^\perp q))$. Since rCq, we have $(rq)^\perp q = (r \wedge q)^\perp \wedge q = (r^\perp \vee q^\perp) \wedge q = r^\perp \wedge q = qr^\perp$. Thus, as p, q, and r commute pairwise, we have $(pq) \wedge ((rq)^\perp q) = (pq) \wedge (qr^\perp) = pqr^\perp$. Likewise, as e, q, and r commute pairwise, we deduce that $(eq) \wedge ((rq)^\perp q) = eqr^\perp$. Thus, it will be sufficient to show that $(pqr^\perp)C(eqr^\perp)$. By Theorem 5.14, $(pr^\perp)C(er^\perp)$; hence, as qCp, $qCr^\perp$, and qCe, we have

$$(pqr^\perp)(eqr^\perp) = q(pr^\perp)(er^\perp q) = q(pr^\perp)(er^\perp)q$$

$$= q(er^\perp)(pr^\perp)q = (eqr^\perp)(pqr^\perp),$$

so $(pqr^\perp)C(eqr^\perp)$. □

5.18 Theorem. *Let* $r := [p, e]$. *Then*: (i) $p_{r^\perp} C e_{r^\perp}$. (ii) $b_{r^\perp} = 0$ *and* $e_{r^\perp} = c^2_{r^\perp} p_{r^\perp} + s^2_{r^\perp} (p_{r^\perp})^{\perp_{r^\perp}}$.

Proof. (i) By Theorem 5.14, $(p \wedge r^\perp)C(e \wedge r^\perp)$, proving (i).

(ii) By Theorem 3.14 (iii) with $q := r^\perp$, the CBS-decomposition of $e_{r^\perp}$ with respect to $p_{r^\perp}$ in $p^\perp A p^\perp$ is $e_{r^\perp} = c^2_{r^\perp} p_{r^\perp} + b_{r^\perp} k_{r^\perp} + s^2_{r^\perp} (p_{r^\perp})^{\perp_{r^\perp}}$. But by (i) and Lemma 3.11, $b_{r^\perp} = 0$. □

5.19 Theorem. $b \le b^{\mathrm{o}} \le [p, e] \le c^{\mathrm{o}} \wedge s^{\mathrm{o}} = (cs)^{\mathrm{o}} = c^{\mathrm{o}} s^{\mathrm{o}}$.

Proof. Put $r := [p, e]$. By Theorem 5.18 (ii), $br^\perp = b_{r^\perp} = 0$, so $b \le r$, and therefore $b \le b^{\mathrm{o}} \le r = [p, e]$.

Put $q := s^{\mathrm{o}}$. Then by Theorem 4.6 (vi), qCp, qCe, and $p \wedge q^\perp = e \wedge q^\perp$, so $(p \wedge q^\perp)C(e \wedge q^\perp)$. Therefore, by Theorem 5.14, $[p, e] \le q = s^{\mathrm{o}}$. A similar argument using Theorem 4.6 (vii) shows that $[p, e] \le c^{\mathrm{o}}$, and we have $[p, e] \le c^{\mathrm{o}} \wedge s^{\mathrm{o}} = (cs)^{\mathrm{o}} = c^{\mathrm{o}} s^{\mathrm{o}}$ (Theorem 4.6 (ii)). □

Using the fact that $b^{\mathrm{o}} \le [p, e]$, we obtain the following alternative characterization of $[p, e]$.

5.20 Theorem. $[p, e]$ *is the smallest projection* v *such that* vCp, vCe, *and* $b^{\mathrm{o}} \le v$.

Proof. Put $r := [p, e]$. By Lemma 5.16 (i), rCp and rCe and by Theorem 5.19, $b^{\mathrm{o}} \le r$. Suppose that $v \in P$, vCp, vCe, and $b^{\mathrm{o}} \le v$. We have to prove that $r \le v$. We have $b \le b^{\mathrm{o}} \le v$, whence $bv^\perp = v^\perp b = 0$. Moreover, as vCp, vCe, and $c^2 = pep + p^\perp e^\perp p^\perp$, it follows that vCc^2. Likewise, vCs^2, whence $v^\perp$ commutes with both e and p, whereas both $v^\perp$ and p commute with c^2, p, s^2, and $p^\perp$. Therefore, by the CBS-decomposition of e with respect to p,

$$ev^\perp = v^\perp e = v^\perp c^2 p + v^\perp bk + v^\perp s^2 p^\perp = v^\perp c^2 p + v^\perp s^2 p^\perp,$$

and since $pv^\perp$ commutes with both $v^\perp c^2 p$ and $v^\perp s^2 p^\perp$ it follows that $pv^\perp$ commutes with $ev^\perp$, i.e., $(p \wedge v^\perp)C(e \wedge v^\perp)$. Consequently, by Theorem 5.14, $r \le v$. □

5.21 Corollary. (i) $b^{\mathrm{o}} = [p, e]$ *iff* eCb^{o}. (ii) *If* $e \in P$, *then* $b^{\mathrm{o}} = [p, e]$ *is the Marsden commutator of the two projections* p *and* e.

Proof. (i) If eCb^{o}, then both $b^{\text{o}}Cp$ and $b^{\text{o}}Ce$ hold, whence $b^{\text{o}} = [p, e]$ by Theorem 5.20. Conversely, by Theorem 5.20 again, if $b^{\text{o}} = [p, e]$, then eCb^{o}.

(ii) Suppose that $e \in P$. Temporarily denoting the Marsden commutator of p and e by $[p, e]_M$, we infer from Remark 5.1 that $b^{\text{o}} = (cs)^{\text{o}} = [p, e]_M$. By [12, Theorem 3.8 (vi)] with $q := e$, we have $eC[p, e]_M$, whence eCb^{o}. Therefore by (i), $[p, e]_M = b^{\text{o}} = [p, e]$. $\square$

The projection p and the effect e are said to be *totally noncompatible* iff $[p, e] = 1$. If p and e are in generic position, then $1 = b^{\text{o}} \leq [p, e]$, and it follows that p and e are totally noncompatible.

5.22 Lemma. *Suppose that p and e are totally noncompatible. Then*: (i) *If $v \in P$, vCp, vCe, and $(p \wedge v)C(e \wedge v)$, then $v = 0$.* (ii) *$c^{\text{o}} = s^{\text{o}} = (cs)^{\text{o}} = 1$.* (iii) *$p \wedge z = p \wedge t = p^{\perp} \wedge z = p^{\perp} \wedge t = 0$.*

Proof. By hypothesis, $[p, e] = 1$. Part (i) follows from Theorem 5.14, part (ii) follows from Theorem 5.19, and (iii) is a consequence of (ii), parts (ii) and (iii) of Theorem 4.6, and De Morgan. $\square$

The following example shows that it is possible to have p and e totally noncompatible (i.e., $[p, e] = 1$), where p and e are not in generic position (i.e., $b^{\text{o}} < [p, e] = 1$).

5.23 Example. Let $\mathbb{R}^3$ be organized as usual into a 3-dimensional real Hilbert space and let A be the synaptic algebra of all self-adjoint linear operators on $\mathbb{R}^3$. Let p_1, p_2, p_3, and p be the (orthogonal) projections onto the one-dimensional subspaces $\{(\alpha, 0, 0) : \alpha \in \mathbb{R}\}$, $\{(0, \beta, 0) : \beta \in \mathbb{R}\}$, $\{(0, 0, \gamma) : \gamma \in \mathbb{R}\}$, and $\{(\xi, \xi, \xi) : \xi \in \mathbb{R}\}$, respectively. Put $e := \frac{1}{4}p_1 + \frac{1}{2}p_2 + \frac{3}{4}p_3$, noting that e is an effect in A and that the set of projections in the spectral resolution of e is $\{0, p_1, p_1 + p_2, 1\}$. As observed above, in forming $[p, e]$, we can omit the projections 0 and 1, whence $[p, e] = [\{p, p_1, p_1 + p_2\}]$. Also, $p_1 + p_2 = p_3^{\perp}$, so by Remark 5.5, $[p, e] = [\{p, p_1, p_3\}]$.

We observe that each of the atoms p_1, p_3, and $p_1^{\perp} \wedge p_3^{\perp} = p_2$ is disjoint from both p and $p^{\perp}$, whence with the notation of Definition 5.3, $p^{d_1} \wedge p_1^{d_2} \wedge p_3^{d_3} = 0$ for all $d \in D^3$. Therefore, by De Morgan,

$$[p, e] = \bigwedge_{d \in D^3} (p^{d_1} \vee p_1^{d_2} \vee p_3^{d_3}) = 1,$$

i.e. p and e are totally noncompatible. In particular $pe \neq ep$. As p is an atom in P, so is $v := kpk$. Thus by Lemma 4.8 (ii), $p \perp v$ and $b^{\text{o}} = p \vee v = p + v$ so

b^{o} is a two-dimensional (i.e., rank 2) projection, and therefore $b^{\text{o}} \neq 1 = [p, e]$. As a consequence (Corollary 5.21), b^{o} does not commute with e.

5.24 Theorem. *Let $r := [p, e]$. Then the projection $p_r = pr = rp = p \wedge r$ and the effect $e_r = er = re = e \wedge r$ are totally noncompatible in rAr.*

Proof. By Lemma 5.16 (i), rCp and rCe, whence, putting $q := r$ in Theorem 5.17, we find that the commutator $[p_r, e_r]_{rAr}$ as calculated in rAr is given by $[p_r, e_r]_{rAr} = r \wedge r = r$. But r is the unit element in rAr, proving the theorem. □

By Theorem 5.18 (i) and Theorem 5.24, the projection $p = p_r + p_{r^\perp}$ and the effect $e = e_r + e_{r^\perp}$ are decomposed into components p_r, e_r that are totally noncompatible in rAr and components $p_{r^\perp}, e_{r^\perp}$ that commute in $r^\perp A r^\perp$.

6 An application of CBS-decomposition

If A is the synaptic algebra of all self-adjoint operators on a complex Hilbert space, then (transcribed to our current notation), T. Morland and S. Gudder prove that, if $e \in E$ and p is an atom in P, then $e \wedge p^\perp$ exists in E [20, Lemma 3.8]. Morland and Gudder's proof uses the Hilbert-space inner product and the Schwarz inequality, and thus is not available for our more general synaptic algebra. However, using the CBS-decomposition we generalize [20, Lemma 3.8] to our present setting in Theorem 6.6 below.

6.1 Standing Assumptions. *In this section the notation and assumptions of Sections 2–5 remain in force. In addition, we assume that* (i) *p is an atom in P and* (ii) *as per Lemma 2.11, $pep = \alpha p$ with $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 1$.*

6.2 Definition. If $\alpha > 0$, we define: (1) $a := \alpha^{-1} bk = \alpha^{-1}(pep^\perp + p^\perp ep) \in A$. (2) $y := p^\perp(1 - a) \in R$. (3) $y^* := (1 - a)p^\perp \in R$.

Provided that $\alpha > 0$, the mapping $f \mapsto yfy^*$ for $f \in A$ is the composition of the quadratic mappings $f \mapsto g := (1 - a)f(1 - a)$ and $g \mapsto p^\perp g p^\perp$, whence it is a linear and order-preserving mapping on A.

We omit the straightforward computational proofs of the next three lemmas.

6.3 Lemma. *Suppose that $\alpha > 0$. Then*: (i) *$ap = p^\perp a = p^\perp ap = \alpha^{-1} p^\perp ep$, $ap^\perp = pa = pap^\perp = \alpha^{-1} pep^\perp$, and $ap + pa = a$.* (ii) *$(ap)^2 = 0$.* (iii) *$(pa)^2 = 0$.* (iv) *$y = p^\perp - ap$ and $y^* = p^\perp - pa$.*

6.4 Lemma. *Suppose that $\alpha > 0$. Then*: (i) *The CBS-decomposition of e with respect to the atom p is $e = \alpha p + \alpha a + s^2 p^{\perp}$.* (ii) $\alpha^2 a^2 = b^2$. (iii) $epe = \alpha^2 p + \alpha^2 a + b^2 p^{\perp}$. (iv) $e - \alpha^{-1} epe = (s^2 - \alpha^{-1} b^2) p^{\perp} = p^{\perp}(s^2 - \alpha^{-1} b^2)$.

6.5 Lemma. *Suppose that $f \in A$ and $\alpha > 0$. Then*: (i) $0 \leq f \leq p^{\perp} \Rightarrow yfy^* = f$. (ii) $0 \leq yey^* = (s^2 - \alpha^{-1} b^2) p^{\perp} = e - \alpha^{-1} epe$.

6.6 Theorem. *The infimum $e \wedge p^{\perp}$ exists in E. In fact, if $\alpha = 0$, then $e \wedge p^{\perp} = e$, and if $\alpha > 0$, then $e \wedge p^{\perp} = (s^2 - \alpha^{-1} b^2) p^{\perp} = e - \alpha^{-1} epe$.*

Proof. If $\alpha = 0$, then $pep = 0$, and as $0 \leq e$ it follows that $pe = ep = 0$ ([3, Axiom SA4]), whence $e \leq p^{\perp}$, so $e = e \wedge p^{\perp}$.

Now suppose that $\alpha > 0$. By Lemma 6.5 (ii), $0 \leq (s^2 - \alpha^{-1} b^2) p^{\perp} = e - \alpha^{-1} epe$. Since $0 \leq epe$, we also have $e - \alpha^{-1} epe \leq e \leq 1$, so $e - \alpha^{-1} epe \in E$. Moreover, $e - \alpha^{-1} epe = (s^2 - \alpha^{-1} b^2) p^{\perp} \leq p^{\perp}$. Suppose that $f \in E$ with $f \leq e, p^{\perp}$. Then by Lemma 6.5 (i), $0 \leq y(e - f)y^* = yey^* - yfy^* = (s^2 - \alpha^{-1} b^2) p^{\perp} - f$, whence $f \leq (s^2 - \alpha^{-1} b^2) p^{\perp}$, and it follows that $e \wedge p^{\perp} = (s^2 - \alpha^{-1} b^2) p^{\perp} = e - \alpha^{-1} epe$. □

6.7 Corollary (Cf. [20, Corollaries 3.9 and 3.10])**.**

(i) *If $p_1, p_2, ..., p_n$ is a finite sequence of mutually orthogonal atoms in P, then $e \wedge (p_1 \vee p_2 \vee \cdots \vee p_n)^{\perp}$ exists in E.*

(ii) *Suppose that every nonzero projection in P is a supremum of a finite sequence of mutually orthogonal atoms in P. Then, for all $q \in P$, the infimum $e \wedge q$ exists in E.*

Proof. (i) The infimum $e \wedge p_1^{\perp}$ exists by Theorem 6.6. Similarly, as $e \wedge p_1^{\perp} \in E$, the infimum $(e \wedge p_1^{\perp}) \wedge p_2^{\perp} = e \wedge (p_1 \vee p_2)^{\perp}$ exists in E. Continuing in this way by induction, we obtain (i).

(ii) Obviously, $e \wedge 1 = e$, so we can assume that $q \neq 1$, whence $q^{\perp} \neq 0$. Therefore by hypothesis, there is a finite sequence $p_1, p_2, ..., p_n$ of mutually orthogonal atoms in P such that $q^{\perp} = p_1 \vee p_2 \vee \cdots \vee p_n$, and it follows from (i) that $e \wedge q$ exists in E. □

The synaptic algebra A is said to be of *rank* r, $r = 1, 2, 3, ...$ iff there are r, but not $r + 1$ mutually orthogonal nonzero projections in P. Clearly, a synaptic algebra of rank r satisfies the hypothesis of Corollary 6.7 (ii). By [6] and [7, Corollary 4.4], a positive-definite spin factor of dimension 2 or more is the same thing as a synaptic algebra of rank 2. Therefore:

6.8 Corollary. *If A is a positive-definite spin factor of dimension 2 or more, $e \in E$, and $q \in P$, then $e \wedge q$ exists in E.*

We note that there are infinite-dimensional positive-definite spin factors.

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